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Genuine electromagnetic wave chaos

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The existence of genuine electromagnetic wave chaos is predicted. The prediction is based on a general nonlinear mechanism that destroys the superposition principle in case the electromagnetic field is allowed to interact dynamically with its boundary. Strong support for this prediction is derived from various model calculations. The proposed chaos mechanism, illustrated here explicitly for the classical Maxwell field, is of general importance for all fields that are allowed to interact dynamically with their boundaries.

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The main point of this paper is to argue that, despite the fact that Maxwell's equations [1] are linear, deterministic chaos [2–4] can occur in the electromagnetic field of a cavity with a movable piston. How is this possible? Is not chaos usually associated with nonlinear systems? The answer is the following. It is true that the Maxwell equations are linear if the electromagnetic field is contained within a cavity with fixed ideally conducting walls. In this case the superposition principle holds and no chaos can emerge. But an electromagnetic field contained inside a cavity exerts a pressure on the cavity walls. Suppose that one of the cavity walls is movable (for instance, the wall at $z = q$ in Fig. 1), and the electromagnetic field is allowed to act on this wall and modify its position in a dynamical way, then the superposition principle is no longer valid and Maxwell's equations inside the cavity are effectively nonlinear. In this case, the electromagnetic field can become truly chaotic in the usual dynamical systems sense [4] showing complex behavior and exponential sensitivity to the initial electromagnetic field configuration. Before launching into a quantitative analysis of the system shown in Fig. 1, we should, at this point, remain some more on the qualitative level and discuss the physical reason why the superposition principle can be violated in as simple a situation as sketched in Fig. 1.

Figure 1 shows a rectangular ideally conducting cavity with side length a in the x direction, side length b in the y direction, and side length q in the z direction. If $q(t) = 0$ for all time, we have a rigid cavity discussed in standard textbooks on electromagnetic theory (see, e.g.,

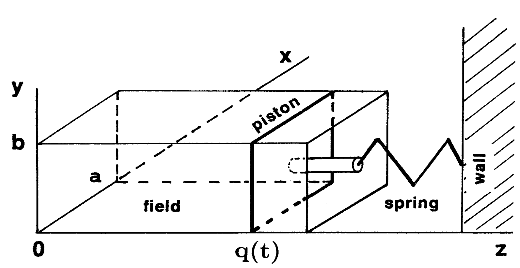

FIG. 1. Sketch of the dynamic cavity.

[1]). This type of cavity does not present any surprises. Its modes can be classified into transverse magnetic (TM) and transverse electric (TE) modes [1], where the z component of the magnetic field, or the z component of the electric field, are identically zero, respectively. But qualitatively new features of an electromagnetic field in a cavity emerge if one of the walls, e.g., the one located at $z = q$, as shown in Fig. 1, is allowed to act like a piston, i.e., it is allowed to slide frictionlessly in the z direction. A spring attached to the piston provides a restoring force. The system shown in Fig. 1, including the electromagnetic field, is closed and autonomous, i.e., the total energy of the system is conserved. In this case any chaos emerging in the system must be self-generated dynamically.

Suppose now that we start the system at some $q(t = 0) = q_0$, $\dot{q}(t = 0) = \dot{q}_0$, and with some initial field configuration inside the cavity denoted by $\vec{E}(x, y, z; t = 0) = \vec{E}_0(x, y, z)$, $\vec{B}(x, y, z; t = 0) = \vec{B}_0(x, y, z)$. The electromagnetic field will exert a pressure on the piston wall. Therefore the piston is subjected not only to the force exerted by the spring, but also to the force produced by the electromagnetic field configuration inside the cavity. As a consequence of the interaction between the electromagnetic field inside the cavity and the dynamics of the piston, we will see a motion of the piston, denoted by $q(t)$, and a corresponding evolution of the electromagnetic field, denoted by $\vec{E}(x, y, z; t)$, $\vec{B}(x, y, z; t)$. Now, if we start with the same initial position of the piston $q(t = 0) = q_0$ and $\dot{q}(t = 0) = \dot{q}_0$, but with a different initial field configuration $\vec{E}(x, y, z; t = 0) = \vec{E}'_0(x, y, z)$, $\vec{B}(x, y, z; t = 0) = \vec{B}'_0(x, y, z)$, then the initial pressure on the piston will in general be different than in the first case, resulting in a *different* piston response function $q'(t) \neq q(t)$. The superposition principle states that, if $\vec{E}(t)$, $\vec{B}(t)$ is a solution of the problem, and $\vec{E}'(t)$, $\vec{B}'(t)$ is also a solution of the problem, then the sum $\vec{E}(t) + \vec{E}'(t)$, $\vec{B}(t) + \vec{B}'(t)$ is also a solution of the problem. This is obviously not the case here, since the components of the sum field [$\vec{E}(t)$, $\vec{B}(t)$ and $\vec{E}'(t)$, $\vec{B}'(t)$, respectively] satisfy different boundary conditions [dictated by $q(t)$ and $q'(t)$, respectively]. Consequently, the sum field does not satisfy any of these conditions and is therefore not a solution. In fact, an electromagnetic field started out with the initial field configuration $\vec{E}_0(x,y,z)+\vec{E}'_0(x,y,z)$, $\vec{B}_0(x,y,z) + \vec{B}'_0(x,y,z)$, negotiating its boundary and time evolution dynamically with the piston, will produce yet a different piston position $q''(t)$, and therefore a dynamical boundary condition not satisfied by either of the two components of the "sum field." Thus, in the case of a dynamical cavity, the superposition principle is violated. This is the central point, and the origin of the possibility of electromagnetic wave chaos in the system shown in Fig. 1. Tiny changes in the initial field configuration in the cavity can give rise to a completely different time evolution of the piston position $q(t)$. Therefore, unlike the situation in a linear system, initial errors do not translate linearly into errors in the final states of the field configurations, but may indeed be exponentially amplified. This prediction is substantiated below with the help of model calculations.

The electromagnetic field in the cavity shown in Fig. 1 satisfies the Maxwell equations [1]

$$\mathrm{div}\vec{E} = 0, \quad \mathrm{div}\vec{B} = 0,$$

$$\mathrm{rot}\vec{B} = \frac{1}{c^2}\partial\vec{E}/\partial t, \quad \mathrm{rot}\vec{E} = -\partial\vec{B}/\partial t. \tag{1}$$

The Maxwell equations (1) have to be solved subject to the boundary conditions [1,5]

$$C_1: \quad \vec{n}\cdot\vec{B} = 0,$$

$$C_2: \quad \vec{n}\times\left[\vec{E}+\vec{v}\times\vec{B}\right] = \vec{0}, \tag{2}$$

where $\vec{n}$ is the unit normal on the boundary and $\vec{v}$ is the speed of the boundary. At the five stationary walls of the cavity the boundary condition C_2 reduces to the static condition

$$C_s: \quad \vec{n}\times\vec{E} = \vec{0}. \tag{3}$$

The solution of (1) subject to C_1 and C_2 is a formidable problem even for the simple setup shown in Fig. 1. The problem is simplified substantially if we replace C_2 by C_s even at the moving boundary of the cavity. In this case the set of Eqs. (1) can be solved with the help of a mode expansion in an adiabatic basis. The change in boundary conditions is not expected to affect the central thesis of this paper: the existence of genuine electromagnetic wave chaos. Moreover, the mode expansion provides us with considerable insight into the workings of the chaos mechanism discussed above.

In this paper we restrict ourselves to the discussion of TM cavity modes only. Following [1] we split the electric and magnetic fields in the cavity into components parallel (E_z, B_z) and transverse ($\vec{E}_t$, $\vec{B}_t$) to the z direction. Taking the mobility of the piston into account, we obtain the following expressions for the TM modes:

$$\vec{E}_t^{(mnp)} = -E_0 A_p^{(mn)}(t)\frac{p\pi}{q\gamma_{mn}^2}\sin(p\pi z/q)\vec{\nabla}_t\psi_{mn},$$

$$E_z^{(mnp)} = E_0 A_p^{(mn)}(t)\cos(p\pi z/q)\psi_{mn},$$

$$\vec{B}_t^{(mnp)} = -\frac{E_0}{\gamma_{mn}^2 c^2}\left[\dot{A}_p^{(mn)}(t)\cos(p\pi z/q) + A_p^{(mn)}(t)\frac{p\pi z}{q^2}\dot{q}\sin(p\pi z/q)\right]\hat{\vec{e}}_z\times\vec{\nabla}_t\psi_{mn},$$

$$B_z^{(mnp)} = 0. \tag{4}$$

The integers $m,n = 1,2,\ldots, p = 0,1,\ldots$ characterize the particular TM mode, E_0 is a constant electric field amplitude, $\hat{\vec{e}}_z$ is a unit vector in the z direction, $A_p^{(mn)}(t)$ is a time dependent amplitude function, c is the speed of light, and $\vec{\nabla}_t$ is the "transverse gradient" given by $\vec{\nabla}_t = (\partial/\partial x, \partial/\partial y)$. The scalar functions ψ_{mn} are functions of x and y only. They are defined by $\psi_{mn}(x,y) = \sin(m\pi x/a)\sin(n\pi y/b)$ and satisfy $[\vec{\nabla}_t^2 + \gamma_{nm}^2]\,\psi_{mn} = 0$, where $\gamma_{mn}^2 = (m\pi/a)^2 + (n\pi/b)^2$. It can be checked that the modes (4) satisfy the boundary conditions C_1 and C_s. The modes (4) are constructed such that the first three Maxwell equations in (1) and the z component of the fourth Maxwell equation are identically satisfied. The transverse component of the rot$\vec{E}$ equation in (1) is satisfied only for $\dot{q}(t)\equiv 0$. In this case the piston is fixed at $z = q$, and (4) represents a collection of stationary TM cavity modes. For $\dot{q}\neq 0$ the motion of the piston couples the TM modes (4). Two cases have to be distinguished. (i) At time $t = 0$ only modes with fixed (m,n) are occupied, whereas the fields and their time derivatives are zero in the modes (m',n') for $m\neq m'$ and $n\neq n'$. In this case the fields in the modes different from (m,n) remain exactly zero for all times. This is the "diagonal" case. Its existence is due to the fact that the piston moves exclusively in the z direction. (ii) If at time $t = 0$ two classes of modes, (m_1,n_1) and (m_2,n_2), with $m_1\neq m_2$ and $n_1\neq n_2$ are occupied, they will be coupled through the moving boundary at $z = q(t)$. If at time $t = 0$ the total energy in the class of modes (m_2,n_2) is close to zero, the energy content in this class can nevertheless be "pumped up" in the course of time due to the motion of the piston driven by the modes in class (m_1,n_1). We call this phenomenon "mechanical mode pumping." In this paper we restrict ourselves to the diagonal case (i) and assume that only modes with fixed (m,n) are occupied initially. We expand the electric and magnetic fields in the cavity according to

$$\mathcal{F}^{(mn)}(x,y,z;t) = \sum_p \mathcal{F}^{(mnp)}(x,y,z;t), \tag{5}$$

where $\mathcal{F}$ stands for any of the components of the electric or the magnetic fields in the cavity. A set of coupled differential equations for the expansion amplitudes $A_p^{(mn)}(t)$ in (4) can be derived from the transverse component of the rot$\vec{E}$ equation in (1). Measuring time in units of $1/(c\gamma_{mn})$ and length in units of $1/\gamma_{mn}$, suppressing the indices m,n, and using the orthogonality of the functions ψ_{mn}, we obtain

$$\ddot{A}_r = \frac{2}{qr}\sum_p\left\{\left(2\frac{\dot{q}^2}{q}-\ddot{q}\right)\Gamma_{rp} - \frac{q}{2}[1+(p\pi/q)^2]\delta_{rp} + \frac{\dot{q}^2}{q}\Omega_{rp}\right\}pA_p - \frac{4\dot{q}}{qr}\Gamma_{rp}p\dot{A}_p. \quad (6)$$

We introduced the coupling matrix elements $\Gamma_{rp} = (r\pi/q^2)\int_0^q \cos(r\pi z/q) z \sin(p\pi z/q)dz$ and $\Omega_{rp} = (rp\pi^2/q^3)\int_0^q \cos(r\pi z/q) z^2 \cos(p\pi z/q)dz$. Measuring energies in units of $E_0^2 ab\epsilon_0/(16\gamma_{mn})$, where ϵ_0 is the electric permittivity of the vacuum, the energy of the electromagnetic field can be expressed as

$$\mathcal{E} = q\sum_r\left\{A_r^2[1+(r\pi/q)^2] + \dot{A}_r^2\right\} + 2\frac{\dot{q}}{q}\sum_{rp}\left\{2\dot{A}_r q(p/r)\Gamma_{rp} + A_r\dot{q}\Lambda_{rp}\right\}A_p, \quad (7)$$

where $\Lambda_{rp} = (rp\pi^2/q^3)\int_0^q \sin(r\pi z/q) z^2 \sin(p\pi z/q)dz$. Since the total energy is conserved, the field energy acts like a potential energy for the piston. Therefore the equation of motion for the piston reads

$$\frac{dP}{dt} = -k(q-\bar{q}) - \frac{\partial\mathcal{E}}{\partial q}, \quad (8)$$

where P is the momentum of the piston, k is the dimensionless spring constant, and $\bar{q}$ is the equilibrium position of the piston in the zero-field case. For all the examples discussed in this paper we use $\bar{q} = 1$. The last term in (8) has to be evaluated according to $(1/\dot{q})d\mathcal{E}/dt$ in order to take the motion of the boundary properly into account. In order to illustrate the emergence of chaos in the electromagnetic field of the dynamic cavity shown in Fig. 1, we will now study a nonrelativistic single-mode model defined by retaining only the TM mode with $p = 1$ and using the momentum-velocity relation $P = \mu\dot{q}$. We choose $\mu = 1$ and $k = 10$. This model describes the field with two amplitudes $A \equiv A_1$ and $\dot{A} \equiv \dot{A}_1$ which act like position and momentum variables. Therefore the phase space of the single-mode model is four dimensional. Since the energy is conserved, the phase-space flow is on a three-dimensional subspace of the four-dimensional phase space. We can visualize the nature of the flow by recording the values of the field variables A and $\dot{A}$ whenever the piston position q passes through $\bar{q}$ with $\dot{q} > 0$. This way we produce a Poincaré surface of section defined by $q = \bar{q}$. Figure 2(a) shows the resulting phase-space portrait for $E = 4$. Because of energy conservation A can only be in the range $-A_{max} < A < A_{max}$, where $A_{max} = \sqrt{E}/\sqrt{1+\pi^2} = 0.606...$. The Poincaré section shown in Fig. 2(a) was generated from a single initial condition with $A_0 = 0.3$, $\dot{A}_0 = 0$, and $q_0 = 1$. The initial piston velocity $\dot{q}_0$ was chosen such that $E = 4$. The coupled equations of motion (6) and (8) were solved with a fourth-order Runge Kutta method [6]. The phase-space portrait in Fig. 2(a) looks chaotic. This result proves that genuine chaos can arise in a linear field theory with a dynamic boundary. But if the field is truly chaotic, we should see exponential sensitivity of the field amplitudes with respect to tiny variations in the initial field configuration.

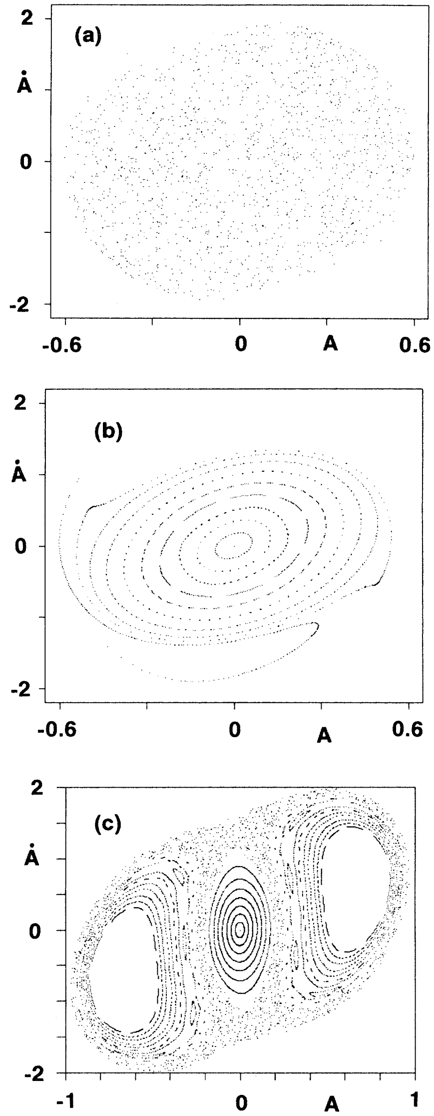

FIG. 2. Poincaré sections for the single-mode model. (a) Nonrelativistic, $E = 4$, $k = 10$, (b) relativistic, $E = 4$, $k = 10$, and (c) relativistic, $E = 20$, $k = 50$.

In order to demonstrate exponential sensitivity in the field, a reference trajectory $A(t), \dot{A}(t), q(t), \dot{q}(t)$ and a trajectory $\tilde{A}(t), \dot{\tilde{A}}(t), \tilde{q}(t), \dot{\tilde{q}}(t)$ were integrated over a time interval of $0 \le t \le 100$. The reference trajectory is identical with the trajectory whose Poincaré section is shown in Fig. 2(a). The "tilde" trajectory was started close to the reference trajectory with initial conditions $\tilde{A}_0 = 0.3 + 10^{-7}$, $\dot{\tilde{A}}_0 = 0$, $\tilde{q}_0 = 1$, and $\dot{\tilde{q}}_0$ such that $E = 4$. The values of the field amplitudes A and $\dot{A}$ ($\tilde{A}$ and $\dot{\tilde{A}}$, respectively) were computed at 1000 equidistant mesh points $t_k = k/10$, $k = 1, ..., 1000$, and the corresponding Euclidean distance $d(t) = \{[A(t)-\tilde{A}(t)]^2 + [\dot{A}(t)-\dot{\tilde{A}}(t)]^2\}^{1/2}$ was computed at the 1000 values t_k. The result is shown in Fig. 3(a). There clearly is an initial exponential rise of

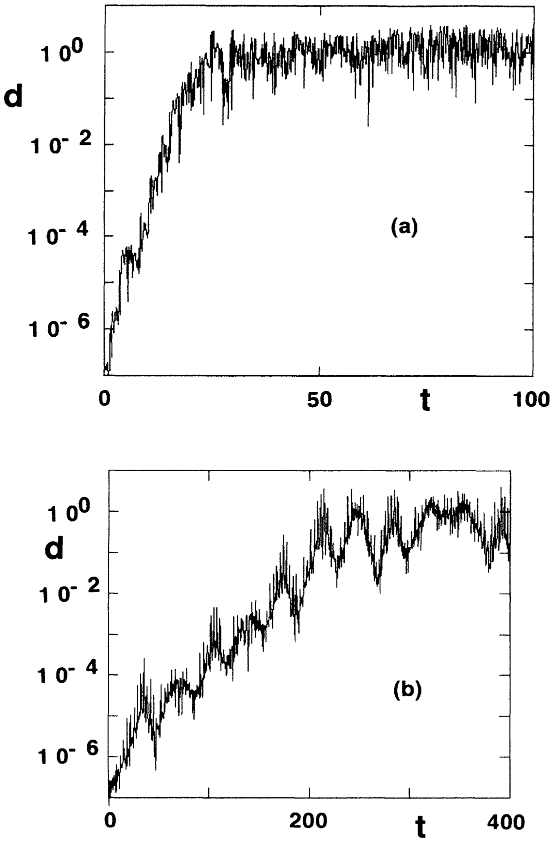

FIG. 3. Euclidean distance $d(t)$ of two initially close trajectories of the single-mode model. (a) Nonrelativistic, $E = 4$, $k = 10$, and (b) relativistic, $E = 20$, $k = 50$.

the distance proving exponential sensitivity upon initial conditions. At $t \approx 25$ the distance saturates at a value that corresponds approximately to the diameter of the dynamically accessible region in the A-$\dot{A}$ phase space.

Inspection of the velocities reveals that $\dot{q}$ is on the order of c. This observation has two consequences. First, the piston motion has to be treated relativistically; second, C_s has to be replaced with C_2 if a realistic computation of the cavity fields is desired. As discussed above, the implementation of C_2 is not possible at present. But it is straightforward to treat the piston motion relativistically. Using $P = \mu\dot{q}/\sqrt{1-\dot{q}^2}$ in (8), we obtain the Poincaré section shown in Fig. 2(b) for energy $E = 4$. The chaos shown in Fig. 2(a) has disappeared. We call this effect relativistic suppression of chaos. Chaos reappears at higher energies. Figure 2(c) shows a Poincaré section for the relativistic model at $E = 20$ and $k = 50$. In this case the phase space is mixed with chaotic and regular regions. Figure 3(b), whose reference trajectory was started in the chaotic regime of Fig. 2(c) with $A = -0.2$, $\dot{A} = 0$ proves that exponential sensitivity persists in the relativistic case.

Chaos and exponential sensitivity also persist if more modes are taken into account. This was checked explicitly in two cases by keeping 5 and 30 modes in the expansion (4). This means that the chaos observed in the single-mode case is not an artifact of the single-mode model, but indicative of the general behavior of electromagnetic fields in a dynamic cavity. It was also found that the energy content of higher modes rapidly decreases with increasing p. Therefore, the piston model exhibits *temporal* chaos, but no electromagnetic *turbulence* since the rapid convergence of the mode expansion indicates that the space degrees of freedom are not chaotic.

In this paper we provided evidence for the existence of genuine electromagnetic wave chaos. The chaos prediction is based on a mechanism that destroys the superposition principle in case the electromagnetic field is allowed to interact dynamically with its boundary. The results of various model calculations support the theoretical prediction. In order to make the calculations more realistic and eventually *prove* the existence of genuine electromagnetic wave chaos one has to pursue two directions. (a) The proper boundary condition C_2 has to be implemented in the computations, and (b) off-diagonal modes (m', n') have to be included in the calculations. Although the diagonal case, i.e., the inclusion of (m, n) modes only, with m and n fixed, is formally exact, the initial presence of the off-diagonal modes cannot be excluded in practice. At finite temperature, e.g., all the modes are at least thermally occupied. Therefore, before investigating point (a), it seems more important to investigate (b) and the phenomenon of mechanical mode pumping.

The chaos mechanism employed in this paper is not restricted to electromagnetic fields. It can be applied generally in any situation where a field is allowed to modify its boundary self-consistently. In all these cases chaos is expected to emerge in certain regimes of system control parameters, irrespective of whether the field is governed by linear or nonlinear field equations. As demonstrated above for some special cases, the signature of chaos in a wave system is sensitive dependence on initial field configurations.

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